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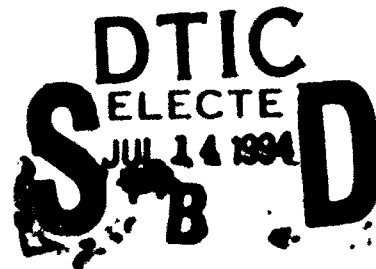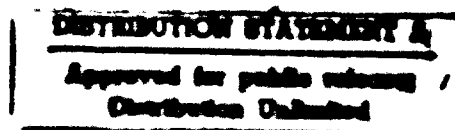
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PSEUDOSPECTRAL COLLOCATION METHODS
FOR FOURTH ORDER DIFFERENTIAL EQUATIONS

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Pseudospectral Collocation Methods for
Fourth Order Differential Equations

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ABSTRACT

Collocation schemes are presented for solving linear fourth order differential equations in one and two dimensions. The variational formulation of the model fourth order problem is discretized by approximating the integrals by a Gaussian quadrature rule generalized to include the values of the derivative of the integrand at the boundary points. Collocation schemes are derived which are equivalent to this discrete variational problem. An efficient preconditioner based on a low-order finite differenc approximation to the same differential operator is presented. The corresponding multi-domain problem is also considered and interface conditions are derived. Pseudospectral approximations which are C^1 continuous at the interfaces are used in each subdomain to approximate the solution. The approximations are also shown to be C^3 continuous at the interfaces asymptotically. A complete analysis of the collocation scheme for the multi-domain problem is provided. The extension of the method to the biharmonic equation in two dimensions is discussed and results are presented for a problem defined in a non-rectangular domain.

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1 Introduction

Spectral methods are characterized by the representation of the solution to a differential equation in terms of a truncated series of smooth global functions which are known as trial or basis functions. The basis functions are usually chosen to be the eigenfunctions of a singular Sturm-Liouville problem (Gottlieb and Orszag, 1977). It is this choice which is responsible for the superior approximation properties of spectral methods over other standard methods of discretization. For linear problems possessing smooth solutions these eigenfunctions yield expansions that converge asymptotically faster than any finite power of N^{-1}.

Two areas of research in spectral methods which are receiving much attention at the current time are the construction and analysis of well-posed approximations to the Stokes and Navier-Stokes equations and the development of methods which can be applied easily to problems defined in complex domains. With respect to the first, it is well-known that in the primitive variable formulation the velocity and pressure approximation spaces need to be compatible to avoid problems of ill-conditioning. This is similar to the Babuška-Brezzi condition required for the corresponding finite element approximation spaces. In two dimensions it is possible to avoid this difficulty by reformulating the governing equations in terms of a stream function. The governing equation is then fourth-order, and nonlinear in the case of the Navier-Stokes equations. In this paper we seek to construct pseudospectral approximations to fourth-order differential equations with the ultimate goal of applying them to solve the nonlinear stream function formulation of the Navier-Stokes equations.

Secondly, the development of techniques for handling complex geometries is essential if spectral methods are to be applied to problems defined in more than just the simplest domains. The basic idea behind domain decomposition is to break up the domain into smaller simpler subdomains in which spectral approximations can be used. The approximations are suitably linked by appropriate interface continuity conditions. The way in which this is implemented is important if the full power of the spectral method, in terms of the accuracy of the approximation, is to be achieved.

In this paper we shall restrict ourselves to the model fourth-order problem in one and two dimensions. Starting from a variational formulation of the problem we shall derive a corresponding collocation problem complete with interface conditions. In a domain decomposition setting this approximation will be chosen to be C^1 continuous implicitly. In addition C^2 continuity across the subdomain boundaries is achieved asymptotically as the order of the approximation is increased.

Although there are many applications of spectral methods to solve second-order elliptic partial differential equations in the literature there is little previous work on fourth-order problems even though the regularity of the solution to these problems is generally higher than for second-order problems. Some interesting ideas are proposed in the works of Morchoisne (1984) and Orszag (1971). Bernardi and Maday (1988) give a survey of strategies that may be employed for fourth-order problems.

Maday and Métivet (1986) have studied Chebyshev spectral and pseudospectral approximations of the stream function formulation of the Navier-Stokes equations. They prove the convergence of the schemes and derive error estimates in weighted Sobolev spaces. Karageorghis and Phillips (1989a,1989b) use a spectral collocation strategy to solve for the laminar

flow through a channel contraction again using a stream function formulation for moderate values of the Reynolds number. They use a domain decomposition method to subdivide the flow region into rectangular subdomains and patching to piece the solutions together, in some sense, across the subdomain interfaces.

In a collocation method the choice of the collocation points is crucial. In spectral methods they are always chosen to be the nodes of a Gaussian quadrature rule principally for two reasons. First, the Lagrange interpolating polynomial which interpolates data at these nodes has good approximation properties. Secondly, the collocation method may be shown to be equivalent to a variational formulation of the problem when the same Gaussian quadrature rule is used to approximate the integrals appearing in this formulation. For second-order problems the Gauss-Lobatto nodes are used because the boundary conditions can then be imposed efficiently. This leads to an optimal error in the resulting spectral approximation (Canuto et al. (1987)). For fourth-order problems two boundary conditions are imposed on the solution. These are usually of Dirichlet and Neumann type. The imposition of these boundary conditions is facilitated by the construction of a generalized Lagrange interpolating polynomial which interpolates the function at the interior nodes and the function and its derivative at the boundary nodes. The generalized Gaussian quadrature rule associated with this interpolating polynomial can then be derived. Quadrature rules of this form are quite well-known in the theory of numerical integration (see, for example, Golub and Kautsky (1983), and the references therein). Golub and Kautsky (1983) describe how the weights in these quadrature rules may be determined computationally. In this paper closed form expressions for the weights are derived using the properties of orthogonal polynomials.

We show that, for fourth-order problems, the natural choice of nodes are the zeros of certain Gegenbauer (or ultraspherical) polynomials. Explicit representations for the quadrature weights are derived for evaluating integrals of the form

$$\int_{-1}^{1} w_\lambda(x) f(x) dx ,$$

where the weight function takes the form

$$w_\lambda(x) = (1 - x^2)^\lambda , \quad \lambda > -1.$$

The particular form of these weights is given when $\lambda = 0$ (the Legendre weight function) and $\lambda = -1/2$ (the Chebyshev weight function). The interior nodes in the case when $\lambda = -1/2$ are the zeros of $T_N''(x)$ whereas the interior Gauss-Chebyshev-Lobatto nodes are the zeros of $T_{N-1}'(x)$.

A collocation scheme for solving a fourth-order model problem is derived by considering a variational formulation of the boundary value problem with suitably defined inner products. The two formulations are shown to be equivalent if the inner product in the discrete variational problem is defined by the generalized Gauss quadrature rule. The linear system of equations which derives from this collocation scheme is ill-conditioned. The condition number of the coefficient matrix scales like $O(N^8)$ where N is the order of the approximation. An efficient preconditioner for this system based on a low order finite difference approximation to the same differential operator is presented. The combination of generalized Gaussian quadrature rules with spectral methods has also been proposed by Bernardi et al. (1990). This idea is extended to multi-domain problems in the present paper. Pseudospectral approximations which are C^1 continuous at the subdomain interfaces are used to

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approximate the solution in each subdomain. The discrete variational problem enables us to derive interface continuity conditions which, in the asymptotic limit, result in C^3 continuous approximations. The variational formulation is used to provide an analysis of the collocation scheme for domain decomposition. The analysis shows that the pseudospectral approximation is optimal in the sense that it is of the same order as the corresponding error in the best approximation. Numerical results are presented in which the usual exponential convergence behaviour of spectral approximations is exhibited. Finally, the extension of the method to two dimensions is described and numerical results presented for a number of model problems. An application of the method to the solution of the biharmonic equation in a non-rectangular domain, an L-shaped region, for which standard spectral methods are not applicable is presented.

2 Variational Formulation of the Model Problem

In this section we consider the variational formulation of the fourth-order model problem. Consider the fourth-order boundary-value problem

$$\frac{d^4u}{dx^4} = f(x), \quad -1 \le x \le 1,$$

(1)

$$u(\pm 1) = 0, \quad \frac{du}{dx}(\pm 1) = 0,$$

where $f(x)$ is a given source function. It is well-known that, for any $f \in H^{-2}(-1,1)$, (1) has a unique solution $u \in H_0^2(-1,1)$ (see Grisvard (1985), for example). A collocation scheme for solving (1) is derived by considering a variational formulation of the problem with suitably defined inner products.

To set up the variational formulation we need to define function spaces for each $\lambda > -1$. Let $L_\lambda^2(-1,1)$ be the Hilbert space defined by

$$L_\lambda^2(-1,1) = \left\{ \begin{array}{l} v : (-1,1) \to \mathbf{R} \text{ is measurable } ; \\ \int_{-1}^{1} w_\lambda(x)v^2(x)dx < \infty \end{array} \right\}$$

endowed with the inner product

$$(u,v)_\lambda = \int_{-1}^{1} w_\lambda(x)u(x)v(x)dx.$$

(2)

We also introduce the Sobolev space $H_\lambda^2(-1,1)$ defined by

$$H_\lambda^2(-1,1) = \{ v : \frac{d^k v}{dx^k} \in L_\lambda^2(-1,1), \ 0 \le k \le 2 \}$$

with corresponding norm

$$\| v \|_{2,\lambda} = \{ \sum_{k=0}^{2} \int_{-1}^{1} w_\lambda(x)(\frac{d^k v}{dx^k})^2 dx \}^{1/2}.$$

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Let $H^2_{\lambda,0}(-1,1)$ be the subspace of $H^2_\lambda(-1,1)$ defined by

$$H^2_{\lambda,0}(-1,1) = \{\, v \in H^2_\lambda(-1,1): \quad v(\pm 1) = 0; \quad v'(\pm 1) = 0 \,\}.$$

Consider now the bilinear form $a_\lambda(.,.)$ defined on

$$H^2_\lambda(-1,1) \times H^2_{\lambda,0}(-1,1)$$

by

$$a_\lambda(u,v) = \int_{-1}^{1} u''(x)[w_\lambda(x)v(x)]'' dx. \tag{3}$$

For any $f \in H^{-2}(-1,1)$ the fourth-order model problem (1) is equivalent to the variational problem : Find $u \in H^2_{\lambda,0}(-1,1)$ such that

$$a_\lambda(u,v) = (f,v)_\lambda, \quad \forall v \in H^2_{\lambda,0}(-1,1). \tag{4}$$

Bernardi and Maday(1991) have proved the following result:

Proposition 2.1 *Let λ satisfy $-1 < \lambda < 1$. The bilinear form a_λ is elliptic on $H^2_{\lambda,0}(-1,1)$ and continuous on $H^2_\lambda(-1,1) \times H^2_{\lambda,0}(-1,1)$, i.e.*

$$\alpha \parallel u \parallel^2_{2,\lambda} \le a_\lambda(u,u), \quad \forall u \in H^2_{\lambda,0}(-1,1), \tag{5}$$

$$\mid a_\lambda(u,v) \mid \le \beta \parallel u \parallel_{2,\lambda} \parallel v \parallel_{2,\lambda}, \quad \forall u \in H^2_\lambda(-1,1), \; v \in H^2_{\lambda,0}(-1,1), \tag{6}$$

respectively, where α and β are positive constants.

This is a generalization of an earlier result of Maday (1990) for $\lambda = -1/2$. An immediate consequence of Proposition 3.1 is the following theorem:

Theorem 2.1 *Let λ satisfy $-1 < \lambda < 1$. For any $f \in H^{-2}_\lambda(-1,1)$ the variational problem (4) has a unique solution $u \in H^2_{\lambda,0}(-1,1)$. Moreover, it satisfies*

$$\parallel u \parallel_{2,\lambda} \le C \parallel f \parallel_{H^{-2}_\lambda(-1,1)}. \tag{7}$$

3 Pseudospectral Approximation

We consider the pseudospectral discretization of the fourth-order problem (1). Let $P_N(-1,1)$ denote the space of algebraic polynomials of degree N or less on the interval $[-1,1]$. Let x_j, $1 \le j \le N-1$ be $N-1$ distinct points in the interval $[-1,1]$ with $x_1 = -1$ and $x_{N-1} = 1$. Throughout this paper we take $N \ge 4$ in order to have at least one point in the interior of the interval $[-1,1]$. Suppose that the values f_j of some function $f(x)$ are given at the points x_j, $1 \le j \le N-1$, together with the values f'_1 and f'_{N-1}, of $f'(x)$ at $x = x_1$, and $x = x_{N-1}$, respectively. To set up the pseudospectral approximation of (1) which automatically satisfies the boundary conditions it is necessary to construct the Lagrange polynomials for this data. Define the polynomials $v(x)$ and $\overline{\Pi}(x)$ by

$$v(x) = (1 - x^2)^2, \qquad \overline{\Pi}(x) = \prod_{j=2}^{N-2}(x - x_j) \tag{8}$$

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It can be verified that the Lagrange polynomial of degree N which interpolates this data is given by

$$p_N(x) = \sum_{j=1}^{N-1} f_j h_j(x) + f_1' \bar{h}_1(x) + f_{N-1}' \bar{h}_{N-1}(x), \qquad (9)$$

where

$$h_j(x) = \begin{cases} \dfrac{\Pi(x)v(x)}{(x-x_j)\Pi'(x_j)v(x_j)} , & 2 \le j \le N-2, \\[3mm] \left[1 - (x-x_j)\dfrac{R_j'(x_j)}{R_j(x_j)}\right] \dfrac{R_j(x)}{R_j(x_j)} , & j = 1, N-1, \end{cases} \qquad (10)$$

$$\bar{h}_j(x) = \frac{\Pi(x)}{\Pi(x_j)} \frac{S_j(x)}{S_j'(x_j)} , \qquad j = 1, N-1, \qquad (11)$$

and

$$R_j(x) = \frac{\Pi(x)v(x)}{(x-x_j)^2} , \qquad S_j(x) = \frac{v(x)}{(x-x_j)} , \qquad j = 1, N-1. \qquad (12)$$

The corresponding integration rule based on these points

$$\int_{-1}^{1} w_\lambda(x) f(x) dx = \sum_{j=1}^{N-1} w_j f(x_j) + \varpi_1 f'(x_1) + \varpi_{N-1} f'(x_{N-1}) , \qquad (13)$$

can be shown to be exact for all $f \in P_{2N-3}(-1,1)$ if the interior nodes x_j, $2 \le j \le N-2$ are chosen to be the zeros of the Gegenbauer or ultraspherical polynomial $G_{N-3}^{(\lambda+2)}$ of degree $N-3$. The location of the nodes are determined by Newton's method and polynomial deflation. For the sake of generality we consider the general case $\lambda > -1$ here although when we investigate the solution of differential equations we will only consider the case $\lambda = 0$. Important properties of the ultraspherical polynomials $G_n^{(\mu)}(x)$ are given in the Appendix (hereinafter the reference (A.m) will be used to denote equation (m) from the Appendix ($m = 1, 2, \ldots$)). The weights depend on λ and this will be assumed in the following.

The polynomials $h_j(x)$, $1 \le j \le N-1$ and $\bar{h}_j(x)$, $j = 1, N-1$ defined by (10) and (11), respectively, form a basis for $P_N(-1,1)$. Therefore, choosing $f(x)$ to be each of these polynomials in turn we obtain explicit expressions for the $N+1$ weights:

$$w_j = \int_{-1}^{1} w_\lambda(x) h_j(x) dx , \qquad 1 \le j \le N-1, \qquad (14)$$

$$\varpi_j = \int_{-1}^{1} w_\lambda(x) \bar{h}_j(x) dx , \qquad j = 1, N-1. \qquad (15)$$

Although only the boundary weights are of relevance to the present paper in that they are required for the statement of the multi-domain collocation problem we give details here of the representations for the interior weights as well. These are necessary if one was to solve the discrete variational problem without restating it as a collocation one. The advantage of doing this is that it results in a symmetric system of linear equations to be solved for the unknown nodal values of the solution.

We are able to derive an original result in which explicit representations for the weights (9,10) are obtained using the properties of the ultraspherical polynomials (see the Appendix).

Let us begin with the weights w_j, $2 \le j \le N-2$, associated with the interior points. The polynomials $\Pi(x)$ and $G_{N-3}^{(\lambda+2)}(x)$ are related by

$$\Pi(x) = \frac{G_{N-3}^{(\lambda+2)}(x)}{A_{N-3}} \, , \qquad (16)$$

since they are of the same degree and have the same zeros, where A_{N-3} is the leading coefficient of $G_{N-3}^{(\lambda+2)}(x)$. Thus using (10) and (14) we may write

$$w_j = \frac{1}{G_{N-3}^{(\lambda+2)'}(x_j)v(x_j)} \int_{-1}^{1} \frac{w_\lambda(x)v(x)G_{N-3}^{(\lambda+2)}(x)}{(x-x_j)}dx, \quad 2 \le j \le N-2. \qquad (17)$$

In order to determine the value of this integral, we make use of the Christoffel-Darboux identity:

$$\sum_{k=0}^{N-3} \frac{G_k^{(\lambda+2)}(x)G_k^{(\lambda+2)}(y)}{\gamma_k} = \frac{G_{N-2}^{(\lambda+2)}(x)G_{N-3}^{(\lambda+2)}(y) - G_{N-3}^{(\lambda+2)}(x)G_{N-2}^{(\lambda+2)}(y)}{\gamma_{N-3}(x-y)}\frac{A_{N-3}}{A_{N-2}}, \qquad (18)$$

where γ_k, $(1 \le k \le N-3)$ is defined by (A.4). Now we replace y by x_j where $G_{N-3}^{(\lambda+2)}(x_j) = 0$ then (18) reduces to

$$\frac{G_{N-2}^{(\lambda+2)}(x_j)A_{N-3}}{\gamma_{N-3}A_{N-2}}\frac{G_{N-3}^{(\lambda+2)}(x)}{(x-x_j)} = -\sum_{k=0}^{N-3} \frac{G_k^{(\lambda+2)}(x)G_k^{(\lambda+2)}(x_j)}{\gamma_k}. \qquad (19)$$

If we now multiply both sides of (19) by $(1-x^2)^{\lambda+2}G_0^{(\lambda+2)}(x)$ and integrate with respect to x over $[-1,1]$ then using the orthogonality property (A.3) we obtain

$$\frac{G_{N-2}^{(\lambda+2)}(x_j)A_{N-3}}{\gamma_{N-3}A_{N-2}} \int_{-1}^{1} \frac{(1-x^2)^{\lambda+2}G_0^{(\lambda+2)}(x)G_{N-3}^{(\lambda+2)}(x)}{(x-x_j)}dx = -G_0^{(\lambda+2)}(x_j),$$

which enables us to write

$$w_j = -\frac{A_{N-2}\gamma_{N-3}}{A_{N-3}v(x_j)G_{N-3}^{(\lambda+2)'}(x_j)G_{N-2}^{(\lambda+2)}(x_j)} \, , \quad 2 \le j \le N-2, \qquad (20)$$

since $G_0^{(\lambda+2)}(x)$ is a constant. Using the recurrence relation (A.6) with $n = N-3$ we write (17) in the form

$$w_j = \frac{2^{2\lambda+5}\Gamma(N+\lambda-1)\Gamma(N+\lambda)}{(N-3)!\Gamma(N+2\lambda+2)} \frac{1}{(1-x_j^2)^2 G_{N-3}^{(\lambda+2)'}(x_j)G_{N-4}^{(\lambda+2)}(x_j)} \, , \qquad (21)$$

for $2 \le j \le N-2$.

Representations for the boundary weights $w_1, w_{N-1}, \bar{w}_1$ and $\bar{w}_{N-1}$ are found using the integrals (A.8) and (A.9). Consider $\bar{w}_1$ which, in view of (11), (12) (15) and (16), may be written in the form

$$\bar{w}_1 = \frac{1}{G_{N-3}^{(\lambda+2)}(-1)S_1'(-1)} \int_{-1}^{1} (1-x^2)^\lambda(1-x)^2(1+x)G_{N-3}^{(\lambda+2)}(x)dx. \qquad (22)$$

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Now $S_1(x) = (1 - x)^2(1 + x)$ and therefore $S_1'(-1) = 4$. The condition (A.5) enables us to write (22) in the form

$$\overline{w}_1 = \frac{(-1)^N(N-3)!\Gamma(\lambda+3)}{4\Gamma(N+\lambda)}\int_{-1}^{1}(1-x)^{\lambda+2}(1+x)^{\lambda+1}G_{N-3}^{(\lambda+2)}(x)dx. \tag{23}$$

The integral in (23) may be evaluated analytically using (A.8) to give

$$\overline{w}_1 = \frac{2^{2\lambda+2}\Gamma(\lambda+2)\Gamma(\lambda+3)(N-3)!}{\Gamma(N+2\lambda+2)}. \tag{24}$$

Similarly we can show that

$$w_1 = \frac{2^{2\lambda+2}\Gamma(\lambda+1)\Gamma(\lambda+3)(N-3)!}{(\lambda+3)\Gamma(N+2\lambda+2)}\{(\lambda+2)N^2 + (\lambda+2)(2\lambda-1)N - (4\lambda^2+9\lambda+3)\}. \tag{25}$$

and also

$$w_{N-1} = w_1 \quad , \quad \overline{w}_{N-1} = -\overline{w}_1. \tag{26}$$

In the special case $\lambda = 0$ the Gegenbauer polynomials coincide with the Legendre polynomials since $w_0(x) = 1$. The boundary weights are given by

$$w_1 = w_{N-1} = \frac{8(2N^2 - 2N - 3)}{3(N-2)(N-1)N(N+1)}, \tag{27}$$

$$\overline{w}_1 = -\overline{w}_{N-1} = \frac{8}{(N-2)(N-1)N(N+1)}, \tag{28}$$

and the interior weights by

$$w_j = \frac{32(N-1)N}{(N-2)(N+1)(N+2)^3}\frac{1}{(1-x_j^2)[G_{N-3}^{(2)}(x_j)]^2}. \tag{29}$$

for $2 \leq j \leq N-2$. This form for the interior weights is derived using (A.6) and (A.10).

When $\lambda = -1/2$, the Gegenbauer polynomials are multiples of the Chebyshev polynomials $T_n(x) = \cos(n\cos^{-1}x)$. In this case the boundary weights are given by

$$w_1 = w_{N-1} = \frac{3\pi(3N^2 - 6N + 1)}{10(N-2)(N-1)N}. \tag{30}$$

$$\overline{w}_1 = -\overline{w}_{N-1} = \frac{3\pi}{4(N-2)(N-1)N}. \tag{31}$$

and the interior weights by

$$w_j = \frac{\pi(N-1)(N-2)^3}{N}\frac{1}{(1-x_j^2)[T_{N-3}''(x_j)]^2}, \quad 2 \leq j \leq N-2. \tag{32}$$

Having written down an expression for a generalized pseudospectral approximation (9) and determined the weights in the corresponding quadrature rule we are now in a position

to write down the discrete problem corresponding to (4). The discrete variational problem corresponding to (4) is : Find $u_N \in P_N(-1,1) \cap H^2_{\lambda,0}(-1,1)$ such that

$$a_\lambda(u_N, v_N) = (f, v_N)_{\lambda,d} , \quad \forall \ v_N \in P_N(-1,1) \cap H^2_{\lambda,0}(-1,1), \tag{33}$$

where the bilinear form $(.,.)_{\lambda,d}$ is defined by

$$(f,g)_{\lambda,d} = \sum_{j=2}^{N-2} w_j f(z_j) g(z_j) + w_1[(fg)(z_1) + (fg)(z_{N-1})] + \overline{w}_1[(fg)'(z_1) - (fg)'(z_{N-1})], \tag{34}$$

and z_j, $2 \leq j \leq N-2$, are the zeros of $G^{(\lambda+3)}_{N-3}(z)$.

Theorem 3.1 *The variational problem (33) is equivalent to the following collocation problem: Find $u_N \in P_N(-1,1) \cap H^2_{\lambda,0}(-1,1)$ such that*

$$u^{(iv)}_N(z_j) = f(z_j) , \quad 2 \leq j \leq N-2, \tag{35}$$

where z_j, $2 \leq j \leq N-2$ are the zeros of $G^{(\lambda+3)}_{N-3}(z)$.

Proof. The left-hand side, $a_\lambda(u_N, v_N)$ is integrated by parts twice to give the following equivalent problem : Find $u_N \in P_N(-1,1) \cap H^2_{\lambda,0}(-1,1)$ such that

$$(u^{(iv)}_N, v_N)_\lambda = (f, v_N)_{\lambda,d} , \tag{36}$$

for all $v_N \in P_N(-1,1) \cap H^2_{\lambda,0}(-1,1)$.

Now a basis for the space $P_N(-1,1) \cap H^2_{\lambda,0}(-1,1)$ are the polynomials $h_j(z)$, $2 \leq j \leq N-2$, defined in (10). These are used as test functions in (36). Since $u^{(iv)}_N v_N \in P_{2N-4}(-1,1) \cap H^2_{\lambda,0}(-1,1)$ and the quadrature rule (13) is exact for all polynomials in $P_{2N-3}(-1,1)$ we have

$$(u^{(iv)}_N, h_j)_\lambda = w_j u^{(iv)}_N(z_j) , \quad 2 \leq j \leq N-2. \tag{37}$$

Further, from the definition (34),

$$(f, h_j)_{\lambda,d} = w_j f(z_j) , \quad 2 \leq j \leq N-2 , \tag{38}$$

and therefore since $w_j > 0$, for $2 \leq j \leq N-2$, we obtain (35) which completes the proof of the theorem.□

We have an analogous result to Theorem 2.1 for the discrete problem:

Theorem 3.2 *Let λ satisfy $-1 < \lambda < 1$. For any $f \in C^0(-1,1)$, the problem (35) has a unique solution $u_N \in P_N(-1,1) \cap H^2_{\lambda,0}(-1,1)$.*

Bernardi and Maday (1991) establish the following error estimate:

Theorem 3.3 *Let λ satisfy $-1 < \lambda < 1$. If the solution u of (29) belongs to $H^\sigma_\lambda(-1,1)$ for a real number $\sigma \geq 1$, and if the data f is such that the function $(1-z^2)^{\frac{3}{2}} f$ belongs to a space $H^\rho_\lambda(-1,1)$ for a real number $\rho \geq 1/2$, the following error estimate between the solutions of (29) and (35) is satisfied*

$$\| u - u_N \|_{2,\lambda} \leq C(N^{2-\sigma} \| u \|_{\sigma,\lambda} + N^{1/2-\rho} \| (1-z^2)^{\frac{3}{2}} f \|_{\rho,\lambda}). \tag{39}$$

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The collocation method (38) results in a system of equations for the values u_j of $u_N(x)$ at the points x_j, $2 \leq j \leq N - 2$. The pseudospectral collocation approximation is then given by

$$u_N(x) = \sum_{j=2}^{N-2} u_j h_j(x) , \qquad (40)$$

where $h_j(x)$ is given by (10) (cf. (9)).

The generalization of the collocation method (35) for problems with inhomogeneous boundary conditions is straightforward. The nature of the pseudospectral approximation (9) is such that inhomogeneous boundary conditions are satisfied exactly by simply inserting the specified values directly into (9). If $H^2_{\lambda,B}(-1,1)$ is the subspace of $H^2_\lambda(-1,1)$ which consists of those functions that satisfy the given inhomogeneous boundary conditions then we have the collocation problem : Find $u_N \in P_N(-1,1) \cap H^2_{\lambda,B}(-1,1)$ such that

$$u_N^{(iv)}(x_j) = f(x_j) , \quad 2 \leq j \leq N - 2,$$

where x_j, $2 \leq j \leq N - 2$ are the zeros of $G_{N-3}^{(\lambda+2)}(x)$.

4   Preconditioning

The collocation problem (35) can be restated in the form of a linear system of algebraic equations

$$A\mathbf{u} = \mathbf{b} \qquad (41)$$

where $\mathbf{u}$ is the vector of values of $u_N(x)$ at the collocation points x_j, $2 \leq j \leq N - 2$, $\mathbf{b}$ is the vector of values of $f(x)$ at these points and A is the $(N - 3) \times (N - 3)$ matrix whose entries are defined by

$$A_{j-1,k-1} = h_k^{(iv)}(x_j), \quad 2 \leq j, k \leq N - 2.$$

The fourth-order pseudospectral differentiation operator A has postive, real eigenvalues. The extreme eigenvalues of A are shown in Table 1. In this table we see that the largest eigenvalue of A scales like N^8 while the smallest eigenvalue is independent of N. Therefore, since the condition number of A is $O(N^8)$ an efficient preconditioner is necessary for the accurate inversion of (41).

Orszag (1980) proposed a preconditioner for spectral methods based on a low-order finite difference approximation to the same differential operator. The advantages of such a preconditioner are that it is sparse, easily invertible and yields an inverse close to the inverse of the original spectral operator. Therefore we propose using a second-order finite difference operator as our preconditioner. This requires the solution of a pentadiagonal system which may be performed very efficiently and stably using Gaussian elimination.

For $3 \leq i \leq N - 3$ the second-order finite difference approximation to $u(x)$ at the point x_i is

$$u^{(iv)}(x_i) \approx a_i u(x_{i-2}) + b_i u(x_{i-1}) + c_i u(x_i) + d_i u(x_{i+1}) + e_i u(x_{i+2})$$

where

$$a_i = \frac{24}{(x_{i-2} - x_i)(x_{i-2} - x_{i-1})(x_{i-2} - x_{i+1})(x_{i-2} - x_{i+2})},$$

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$$b_i = \frac{-24}{(x_{i-1} - x_i)(x_{i-2} - x_{i-1})(x_{i-1} - x_{i+1})(x_{i-1} - x_{i+2})},$$

$$d_i = \frac{-24}{(x_i - x_{i+1})(x_{i-2} - x_{i+1})(x_{i-1} - x_{i+1})(x_{i+1} - x_{i+2})},$$

$$e_i = \frac{24}{(x_i - x_{i+2})(x_{i-2} - x_{i+2})(x_{i-1} - x_{i+2})(x_{i+1} - x_{i+2})},$$

$$c_i = -(a_i + b_i + d_i + e_i).$$

A similar formula holds at x_2 and x_{N-2} after taking into account the homogeneous Neumann boundary conditions. Let H denote the finite difference differentiation matrix defined by the above equations. We are interested in the eigenvalue spectrum of the operator $H^{-1}A$ since this governs the rate of convergence of the preconditioned iterative method for solving (41). The eigenvalues of $H^{-1}A$ are real and positive. The extreme eigenvalues of $H^{-1}A$ are shown in Table 2. Again the smallest eigenvalue remains independent of the choice of N while the largest eigenvalue grows very slowly with N. The entries in this table demonstrate the effectiveness of H as a preconditioner for A. Haldenwang et al (1984) showed theoretically that the eigenvalues of the corresponding preconditioned second-order pseudospectral differentiation operator lie between 1 and $(\pi/2)^2$. From this result one would expect that the eigenvalues in the case of the fourth-order problem to lie between 1 and $(\pi/2)^4$. We can see from Table 2 that they do indeed lie between these bounds.

5 Analysis of the Multi-Domain Problem

Given a fixed integer M we consider a partition of $(-1, 1)$ into M subintervals I_m where

$$I_m = (d_m, d_{m+1}),$$

and the d_m are $M + 1$ points in $(-1, 1)$ such that

$$-1 = d_0 < d_1 < \cdots < d_{M-1} < d_M = 1.$$

Associated with each subinterval I_m, we define a set points x_j^m, $1 \leq j \leq N - 1$, and weights w_j^m, $1 \leq j \leq N-1$, $\overline{w}_j$, $j = 1, N-1$, which correspond to a generalized Gaussian quadrature rule of the form (13) defined on I_m. Let $h_j^m(x)$; $1 \leq j \leq N - 1$, and $\overline{h}_j^m(x)$, $j = 1, N - 1$, be the corresponding interpolating functions which have compact support on the interval I_m. We introduce the finite dimensional spaces

$$Y_N = \{\phi \in L^2(-1, 1) : \phi \mid_{I_m} \in P_N(I_m)\},$$

where N is some integer and $P_N(\Lambda)$ denotes the set of all polynomials of degree less than or equal to N over Λ. In order to discretize the space $H_0^2(-1, 1)$, let us introduce the spaces

$$X_N = Y_N \cap H_0^2(-1, 1),$$

$$Z_N = \{\phi \in L^2(-1, 1) : \phi \mid_{I_m} \in P_N(I_m) \cap H_0^2(I_m)\}.$$

The elements of X_N are continuous and have continuous derivatives at the points d_m, $1 \leq m \leq M - 1$, and vanish along with their first derivatives at $x = \pm 1$.

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In this paper we shall only consider the case $\lambda = 0$ as far as domain decomposition is concerned. This is the only value of λ for which the weight function over $(-1,1)$ is the same as the weight function over each of the subintervals I_m, $1 \leq m \leq M$. Throughout this section, in which $\lambda = 0$, the zero subscript has been deleted. For example, $a(.,.)$ is used synonymously with $a(.,.)_0$ and the corresponding norm notation has also been altered accordingly. We now define the discrete problem: Find $u_N \in X_N$ such that

$$a(u_N, v_N) = (f, v_N)_M, \ \forall \, v_N \in X_N, \tag{42}$$

where the bilinear form $(.,.)_M$ is defined by

$$(f,g)_M = \sum_{m=1}^{M} (f,g)_m, \tag{43}$$

where

$$(f,g)_m = \sum_{j=2}^{N-2} w_j^m f(x_j^m) g(x_j^m) + w_1^m [(fg)(x_1^m) + (fg)(x_{N-1}^m)] + \overline{w}_1^m [(fg)'(x_1^m) - (fg)'(x_{N-1}^m)].$$

Lemma 5.1 *For any real number $\sigma \geq 2$ and for any $\phi \in H_0^2(-1,1) \cap H^\sigma(-1,1)$ we have*

$$\| \phi - \pi_{N,0}^2 \phi \|_2 \leq C N^{2-\sigma} \| \phi \|_\sigma,$$

where $\pi_{N,0}^2$ is the orthogonal projection operator from $H_0^2(-1,1)$ onto $P_N(-1,1) \cap H_0^2(-1,1)$.

Proof. See Bernardi and Maday (1991). $\Box$

Since $H^2(-1,1)$ is contained in $C^1([-1,1])$ we can show that for any $\phi \in H^2(-1,1)$, there exists a cubic polynomial ϕ_0 such that $\phi - \phi_0 \in H_0^2(-1,1)$ and for any real number $s \geq 0$,

$$\| \phi_0 \|_s \leq C \| \phi \|_2 .$$

Now define an operator π_N^2 by $\pi_N^2 \phi = \pi_{N,0}^2 (\phi - \phi_0) + \phi_0$ from $H^2(-1,1)$ onto $P_N(-1,1)$. So that if $\phi \in H^\sigma(-1,1)$ then by Lemma 4.1,

$$
\begin{aligned}
\| \phi - \pi_N^2 \phi \|_2 &= \| (\phi - \phi_0) - \pi_{N,0}^2 (\phi - \phi_0) \|_2 \\
&\leq C N^{2-\sigma} \| \phi - \phi_0 \|_\sigma \\
&\leq C N^{2-\sigma} \| \phi \|_\sigma .
\end{aligned}
$$

We can easily verify that this operator satisfies

$$(\pi_N^2 \phi)(\pm 1) = \phi(\pm 1), (\pi_N^2 \phi)'(\pm 1) = \phi'(\pm 1).$$

Theorem 5.1 *There exists an operator $\tilde{\pi}_N^2$ from $H_0^2(-1,1)$ onto X_N satisfying*

$$\| \psi - \tilde{\pi}_N^2 \psi \|_2 \leq C N^{2-\sigma} \| \psi \|_\sigma, \tag{44}$$

for any function $\psi \in H^\sigma(-1,1) \cap H_0^2(-1,1)$ with $\sigma \geq 2$.

Proof. We recall that for a general interval (a, b) there exists a projection operator π_N from $H^2(a, b)$ onto $P_N(a, b)$ satisfying

$$\| w - \pi_N w \|_{H^2(a,b)} \le C N^{2-\sigma} \| w \|_{H^\sigma(a,b)}, \tag{45}$$

$$\pi_N w(a) = w(a), \quad (\pi_N w)'(a) = w'(a), \tag{46}$$

$$\pi_N w(b) = w(b), \quad (\pi_N w)'(b) = w'(b), \tag{47}$$

for all $w \in H^\sigma(a, b)$.

Let us define the projection operators $\pi_{N,m}$, for $1 \le m \le M$, as being the projection operators from $H^2(I_m)$ onto $P_N(I_m)$. We deduce that the element $\tilde{\pi}_N^2 \psi$ defined on each I_m by

$$\tilde{\pi}_N^2 \psi(x) = \pi_{N,m} \psi(x), \quad \forall x \in I_m,$$

is an element of $P_N(-1, 1) \cap H_0^2(-1, 1)$ that satisfies due to (48)

$$\| \psi - \tilde{\pi}_N^2 \psi \|_2 \le C N^{2-\sigma} \| \psi \|_\sigma. \qquad \square \tag{48}$$

Define $J_N f$ to be the Lagrange interpolating polynomial which interpolates the function f at the $N - 3$ interior collocation points of the generalized Gauss quadrature rule on $(-1, 1)$. Then Bernardi and Maday (1991) have shown that

Lemma 5.2 *For any real number $\rho > 1/2$ and for any ϕ such that the function $(1 - x^2)^{\frac{1}{2}} \phi \in H^\rho(-1, 1)$, the following inequality holds*

$$\| (1 - x^2)^{\frac{1}{2}} (f - J_N f) \|_0 \le C N^{1/2-\rho} \| (1 - x^2)^{\frac{1}{2}} f \|_\rho. \tag{49}$$

Lemma 5.3 *For any real number $\rho_m > 1/2$ and for any f such that the function $[(d_{m+1} - x)(x - d_m)]^{1/2} f \in H^{\rho_m}(I_m)$, then*

$$\sup_{v_N \in P_N(I_m) \cap H_0^2(I_m)} \frac{(f, v_N)_{I_m} - (f, v_N)_m}{\| v_N \|_{H^2(I_m)}} \le C N^{1/2-\rho_m} \| [(d_{m+1} - x)(x - d_m)]^{3/2} f \|_{H^{\rho_m}(I_m)}, \tag{50}$$

where $(.,.)_{I_m}$ is the L^2 inner product on I_m.

Proof. The generalized Gauss quadrature rule on I_m is exact for any polynomial in $P_{2N-3}(I_m)$ and so for any $v_N \in P_N(I_m) \cap H_0^2(I_m)$ we have

$$(f, v_N)_{I_m} - (f, v_N)_m = (f - J_N f, v_N)_{I_m}.$$

where J_N is the Lagrange interpolation operator at the $N - 3$ interior nodes of a generalized Gauss rule on the interval I_m. We recall that the mapping $w \mapsto w/[(d_{m+1} - x)(x - d_m)]^2$ is continuous from $H_0^2(I_m)$ into $L^2(I_m)$. Then we can write

$$(f, v_N)_{I_m} - (f, v_N)_m \le C \| [(d_{m+1} - x)(x - d_m)]^{3/2} (f - J_N f) \|_{L^2(I_m)} \| v_N \|_{H^2(I_m)}.$$

Finally using Lemma 4.2 we obtain

$$(f, v_N)_{I_m} - (f, v_N)_m \le C N^{1/2-\rho_m} \| [(d_{m+1} - x)(x - d_m)]^{3/2} f \|_{H^{\rho_m}(I_m)} \| v_N \|_{H^2(I_m)}, \tag{51}$$

from which we deduce the result. $\square$

Theorem 5.2 *Let us suppose that the solution u to (32) belongs to $H^\sigma(-1,1)$, for a real number $\sigma \geq 2$ and that $[(d_{m+1} - x)(x - d_m)]^2 f \in H^{\rho_m}(I_m)$ for a real number $\rho_m > 1/2$ for each $m = 0, 1, \dots, M - 1$. Then the following error estimate holds:*

$$\| u - u_N \|_2 \leq C(N^{2-\sigma} \| u \|_\sigma + \sum_{m=0}^{M-1} N^{1/2-\rho_m} \| [(d_{m+1} - x)(x - d_m)]^{3/2} f \|_{H^{\rho_m}(I_m)}). \quad (52)$$

Proof. Let us define $u^* = u - u^0$ and $u_N^* = u_N - u_N^0$, where u^0 and u_N^0 are piecewise cubic polynomials such that $u^* |_{I_m} \in H_0^2(I_m)$ and $u_N^* |_{I_m} \in P_N(I_m) \cap H_0^2(I_m)$, $0 \leq m \leq M - 1$. Then Proposition 2.1, together with (4) and (42), gives for any $v_N \in Z_N$,

$$\begin{aligned}
\| u_N^* - v_N \|_2^2 &\leq C(a(u_N^* - v_N, u_N^* - v_N) \\
&= C(a(u^* - v_N, u_N^* - v_N) - (f, u_N^* - v_N) + (f, u_N^* - v_N)_M), \quad (53)
\end{aligned}$$

from which we obtain

$$\| u^* - u_N^* \|_2 \leq C \left\{ \inf_{v_N \in Z_N} \| u^* - v_N \|_2 + \sup_{w_N \in Z_N} \frac{(f, w_N) - (f, w_N)_M}{\| w_N \|_2} \right\}. \quad (54)$$

We choose $v_N = \bar{\pi}_N^2 u^*$ and use Theorem 4.1 to show that

$$\inf_{v_N \in Z_N} \| u^* - v_N \|_2 \leq C N^{2-\sigma} \| u^* \|_\sigma, \quad \sigma \geq 2. \quad (55)$$

Since $w_N \in Z_N$ we have on each interval I_m

$$(f, w_N)_{I_m} - (f, w_N)_m = (f - J_N f, w_N)_{I_m}.$$

We can also show that

$$\sup_{w_N \in Z_N} \frac{(f, w_N) - (f, w_N)_M}{\| w_N \|_2} \leq \sum_{m=0}^{M-1} \sup_{w_N \in P_N(I_m) \cap H_0^2(I_m)} \frac{(f, w_N)_{I_m} - (f, w_N)_m}{\| w_N \|_{H^2(I_m)}},$$

and therefore using Lemma 4.3 we may deduce that

$$\sup_{w_N \in Z_N} \frac{(f, w_N) - (f, w_N)_M}{\| w_N \|_2} \leq C \sum_{m=0}^{M-1} N^{1/2-\rho_m} \| [(d_{m+1} - x)(x - d_m)]^{3/2} f \|_{H^{\rho_m}(I_m)}. \quad (56)$$

Since

$$\| u - u_N \|_2 \leq \| u^* - u_N^* \|_2 + \| u^0 - u_N^0 \|_2$$

and

$$\| u^0 - u_N^0 \|_2 \leq C \| u^* - u_N^* \|_2,$$

then

$$\| u - u_N \|_2 \leq C \| u^* - u_N^* \|_2.$$

Finally using (54)-(56) we obtain the result. $\square$

We now set up the collocation scheme for the domain decomposition problem. We define $u_N \in X_N$ which interpolates data at the points x_j^m, $1 \leq j \leq N - 1$, $1 \leq m \leq M$ by

$$u_N(x) = \sum_{j=1}^{N-1} u_j^m h_j^m(x) + (u')_1^m \overline{h}_1^m(x) + (u')_{N-1}^m \overline{h}_{N-1}^m(x), \quad x \in I_m, \quad (57)$$

where

$$u_N^m = u_1^{m+1}, \quad (u')_N^m = (u')_1^{m+1}, \; 1 \le m \le M-1. \tag{58}$$

Theorem 5.3 *The variational problem (42) with the discrete inner product defined by (43) is equivalent to the following collocation problem : Find $u_N \in X_N$ such that*

$$u_N^{iv}(z_j^m) = f(z_j^m), \; 2 \le j \le N-2, \; 1 \le m \le M, \tag{59}$$

$$u_N''(x_1^{m+1}+) - u_N''(x_{N-1}^m-) = w_1^{m+1} r(x_1^{m+1}+) + w_{N-1}^m r(x_{N-1}^m-),$$
$$1 \le m \le M-1, \tag{60}$$

$$u_N'''(x_1^{m+1}+) - u_N'''(x_{N-1}^m-) = -w_1^{m+1} r(x_1^{m+1}+) - w_{N-1}^m r(x_{N-1}^m-)$$
$$-\overline{w}_1^{m+1} r'(x_1^{m+1}+) - \overline{w}_{N-1}^m r'(x_{N-1}^m-),$$
$$1 \le m \le M-1, \tag{61}$$

and where $r(x) = u_N^{iv}(x) - f(x)$.

Proof. Let us examine $a(u_N, v_N)$ defined by (3). By linearity we may write the integral on the right-hand side of (3) as the sum of integrals over each subinterval I_m for $1 \le m \le M$. Subsequent integration-by-parts twice gives

$$a(u_N, v_N) = \sum_{m=1}^M \int_{I_m} u_N^{iv}(x) v_N(x) dx - \sum_{m=1}^{M-1} \{[u_N'' v_N'](x_{N-1}^m) - [u_N''' v_N](x_{N-1}^m)\}, \tag{62}$$

where $[f](y) \equiv f(y+) - f(y-)$ denotes the jump at $x = y$ in f.

We choose as our basis for the space X_N the polynomials $h_j^m(x)$, $2 \le j \le N-2$, $1 \le m \le M-1$ and $h_{N-1}^m(x)$, $\overline{h}_{N-1}^m(x)$, $1 \le m \le M-1$. The use of these polynomials as test functions in (42) with the discrete inner product given by (43) results in (59)-(61) which completes the proof of the theorem. $\square$

Remark 1 *Note that in view of the expressions for the weights given in (27) and (28),*

$$w_1 = w_N = O(N^{-2}), \quad \overline{w}_1 = -\overline{w}_N = O(N^{-4}), \quad as \; N \to \infty,$$

and therefore from (60) and (61) we can write

$$u_N''(x_1^{m+1}+) - u_N''(x_{N-1}^m-) = O(N^{-2}),$$

$$u_N'''(x_1^{m+1}+) - u_N'''(x_{N-1}^m-) = O(N^{-2}),$$

as $N \to \infty$. Thus we have second and third order continuity at the interface asymptotically, as $N \to \infty$.

6 The Biharmonic Problem in Two Dimensions

Consider the biharmonic problem

$$\nabla^4 \psi(x,y) = f(x,y) , \quad \text{in } \Omega , \tag{63}$$

$$\psi(x,y) = g_1(x,y) , \qquad \text{on } \Gamma , \tag{64}$$

$$\frac{\partial \psi}{\partial n}(x,y) = g_2(x,y) , \quad \text{on } \Gamma , \tag{65}$$

where $\Omega = (-1,1) \times (-1,1)$ and Γ is the boundary of Ω. Grisvard (1985) shows that provided the boundary data satisfies certain compatibility conditions there exists $\psi^b \in H^2(\Omega)$ satisfying (64) and (65). Since we are primarily concerned with the domain d   position problem we only consider the case when the weight function is unity. The a$\cdots$ for the single domain problem is thus greatly simplified.

In order to write down the variational formulation of the problem (63)-(65) we define the bilinear form on $H^2(\Omega) \times H^2(\Omega)$:

$$a(\psi, \phi) = \iint_\Omega (\nabla^2 \psi)(\nabla^2 \phi) dx dy . \tag{66}$$

The biharmonic problem (63)-(65) is then equivalent to the following variational problem: Find $\psi \in H^2(\Omega)$ such that $(\psi - \psi^b) \in H_0^2(\Omega)$ and

$$a(\psi, \phi) = (f, \phi) , \quad \text{for all } \phi \in H_0^2(\Omega) , \tag{67}$$

where

$$(f, \phi) = \iint_\Omega f\phi \, dx dy .$$

We see that ψ is a solution of the variational problem (67) if and only if $\hat{\psi} = \psi - \psi^b$ is a solution of the problem: Find $\hat{\psi} \in H_0^2(\Omega)$ such that

$$a(\hat{\psi}, \phi) = (f, \phi) - a(\psi^b, \phi) , \quad \text{for all } \phi \in H_0^2(\Omega) . \tag{68}$$

It can be easily verified that the bilinear form $a(.,.)$ defined by (63) is continuous and elliptic on $H_0^2(\Omega) \times H_0^2(\Omega)$ and hence that problem (71) has a unique solution in $H_0^2(\Omega)$ for $f \in H^{-2}(\Omega)$.

Let $P_N(\Omega)$ be the space of algebraic polynomials of degree at most N in each co-ordinate direction. The collocation problem associated with (63)-(65) is:
Find $\psi_N \in P_N(\Omega) \cap H^2(\Omega)$ such that

$$\nabla^4 \psi_N(x,y) = f(x,y) , \qquad (x,y) \in R , \tag{69}$$

$$\psi_N(x,y) = g_1(x,y) , \qquad (x,y) \in S \cup T , \tag{70}$$

$$\frac{\partial \psi_N}{\partial n}(x,y) = g_2(x,y) , \quad (x,y) \in S \cup T , \tag{71}$$

$$\frac{\partial^2 \psi_N}{\partial n \partial t}(x,y) = \frac{\partial g_2}{\partial t}(x,y) , \qquad (x,y) \in T , \tag{72}$$

where $\partial/\partial n$ and $\partial/\partial t$ represent differentiation normal and tangential to Γ, respectively, the sets R, S and T are defined by

$$R = \{(\xi_i, \xi_j) : 2 \le i, j \le N - 2\},$$
$$S = \{(\xi_i, \pm 1), (\pm 1, \xi_i) : 2 \le i \le N - 2\},$$
$$T = \{(\pm 1, \pm 1)\},$$

and $G^{(2)}_{N-3}(\xi_i) = 0$, $2 \le i \le N - 2$. There are a total of $(N + 1)^2$ linear equations for the $(N + 1)^2$ unknowns. The dimension of $P_N(\Omega)$ is $(N + 1)^2$. The basis functions in $2D$ are the tensor product of the one-dimensional basis functions given by (10) and (11).

We define the two-dimensional discrete inner product in an analogous way to (34) by applying the quadrature rule in each co-ordinate direction. So in the case when one of the functions ψ or ϕ belongs to $H_0^2(\Omega)$ we have

$$(\psi, \phi)_N = \sum_{i=2}^{N-2} \sum_{j=2}^{N-2} w_i w_j \psi(\xi_i, \xi_j) \phi(\xi_i, \xi_j) . \tag{73}$$

Next we define the discrete bilinear form $a_N(.,.)$ by

$$a_N(\psi, \phi) = (\nabla^4 \psi, \phi)_N . \tag{74}$$

Theorem 6.1 *If there is a function $\psi_N^b \in P_N(\Omega) \cap H^2(\Omega)$ satisfying the boundary conditions (67)-(68) then the collocation problem (69)-(72) is equivalent to the variational problem:*
Find $\psi_N \in P_N(\Omega) \cap H^2(\Omega)$ such that $(\psi_N - \psi_N^b) \in H_0^2(\Omega)$ and

$$a_N(\psi_N, \phi_N) = (f, \phi_N)_N , \quad \text{for all } \phi_N \in P_N(\Omega) \cap H_0^2(\Omega) . \tag{75}$$

Proof. On each horizontal or vertical side of Ω, ψ_N and ψ_N^b are polynomials of degree N satisfying $N + 1$ conditions and so are identical on Γ. The same argument applies to their normal derivatives and so $(\psi_N - \psi_N^b) \in P_N(\Omega) \cap H_0^2(\Omega)$. If we now choose $\phi_N(x, y) = h_j(x) h_k(y)$, $2 \le j, k \le N - 2$, then (75) implies (69)-(72). Conversely, since these $(N - 3)^2$ polynomials form a basis for $P_N \cap H_0^2(\Omega)$, (69)-(72) implies (75). □

Let us now turn our attention to the problem of domain decomposition, and for simplicity restrict ourselves for the moment to the case when Ω is divided into two subdomains with interface

$$\gamma = \{(0, y) : -1 \le y \le 1\} .$$

We define

$$\Omega_1 = \{(x, y) : -1 \le x \le 0, -1 \le y \le 1\} ,$$
$$\Omega_2 = \{(x, y) : 0 \le x \le 1, -1 \le y \le 1\} ,$$

and Γ_k is the boundary of Ω_k for $k = 1, 2$. Define the subspace V of $H^2(\Omega_1) \times H^2(\Omega_2)$ by

$$V = \{\Psi = (\psi^1, \psi^2) \in H^2(\Omega_1) \times H^2(\Omega_2) : \psi^1 = \psi^2, \frac{\partial \psi^1}{\partial x} = \frac{\partial \psi^2}{\partial x} \text{ on } \gamma\} ,$$

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and the subspace V_0 of V by

$$V_0 = \{ \Psi = (\psi^1, \psi^2) \in V \ : \ \psi^m = \frac{\partial \psi^m}{\partial n} = 0 \text{ on } \Gamma \text{ for } m = 1, 2 \} \ .$$

We consider the bilinear form defined on $V \times V$ by

$$a(\Psi, \Phi) = a_1(\psi^1, \phi^1) + a_2(\psi^2, \phi^2) \ , \tag{76}$$

where

$$a_k(\psi^k, \phi^k) = \iint_{\Omega_k} (\nabla^2 \psi^k)(\nabla^2 \phi^k) dx dy \ ,$$

We can show, using Green's theorem, that if $\Phi \in V_0$ then the above bilinear form may be written as

$$a(\Psi, \Phi) = \iint_{\Omega_1} (\nabla^4 \psi^1) \phi^1 dx dy + \iint_{\Omega_2} (\nabla^4 \psi^2) \phi^2 dx dy$$

$$+ \int_\gamma \frac{\partial^2}{\partial x^2}(\psi^1 - \psi^2) \frac{\partial \phi^1}{\partial x} dy - \int_\gamma \frac{\partial^3}{\partial x^3}(\psi^1 - \psi^2) \phi^1 dy \ . \tag{77}$$

If there exists $\Psi^b \in V$ satisfying (64)-(65) then the variational problem is: Find $\Psi \in V$ such that $\Psi - \Psi^b \in V_0$ and

$$a(\Psi, \Phi) = \iint_{\Omega_1} f^1 \phi^1 dx dy + \iint_{\Omega_2} f^2 \phi^2 dx dy \ , \tag{78}$$

where f^k is the restriction of f to Ω_k.

The variational problem (78) is equivalent to the following interface problem:

$$\nabla^4 \psi^k = f^k \ , \quad \text{in } \Omega_k \ , \quad k = 1, 2 \ , \tag{79}$$

$$\psi^k = g_1 \ , \quad \text{on } \Gamma \cap \Gamma_k \ , \quad k = 1, 2 \ , \tag{80}$$

$$\frac{\partial \psi^k}{\partial n} = g_2 \ , \quad \text{on } \Gamma \cap \Gamma_k \ , \quad k = 1, 2 \ , \tag{81}$$

$$\psi^1 = \psi^2 \ , \quad \frac{\partial \psi^1}{\partial x} = \frac{\partial \psi^2}{\partial x} \ , \quad \text{on } \gamma \ . \tag{82}$$

Define the finite dimensional space V_N by

$$V_N = \{ \Psi = (\psi^1, \psi^2) \in P_N(\Omega_1) \cap H^2(\Omega_1) \times P_N(\Omega_2) \cap H^2(\Omega_2) :$$

$$\psi^1 = \psi^2 \ , \quad \frac{\partial \psi^1}{\partial x} = \frac{\partial \psi^2}{\partial x} \text{ on } \gamma \} \ ,$$

and the subspace $V_{N,0}$ of V_N by

$$V_{N,0} = \{ \Psi = (\psi^1, \psi^2) \in V_N \ : \ \psi^m = \frac{\partial \psi^m}{\partial n} = 0 \text{ on } \Gamma \text{ for } m = 1, 2 \} \ .$$

In the case when one of Ψ or Φ belong to $V_{N,0}$ we define a discrete inner product by

$$(\Psi, \Phi)_N = (\psi^1, \phi^1)_N^1 + (\psi^2, \phi^2)_N^2 \ ,$$

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where

$$(\psi^k, \phi^k)_N^k = \sum_{i=2}^{N-2} \sum_{j=2}^{N-2} w_i^k w_j \psi^k(\xi_i^k, \xi_j) \phi^k(\xi_i^k, \xi_j)$$

$$+ \sum_{j=2}^{N-2} w_{N-1}^k w_j \psi^k(0, \xi_j) \phi^k(0, \xi_j)$$

$$+ (-1)^{k+1} \sum_{j=2}^{N-2} \overline{w}_{N-1}^k w_j [\frac{\partial}{\partial n}(\psi^k \phi^k)]_{(0,\xi_j)} ,$$

for $k = 1, 2$, where $\partial/\partial n$ is the normal derivative to Ω_k. The discrete bilinear form on $V_N \times V_N$ is defined by

$$a_N(\Psi, \Phi) = a_N^1(\psi^1, \phi^1) + a_N^2(\psi^2, \phi^2) ,$$

where

$$a_N^k(\Psi^k, \Phi^k) = (\nabla^4 \psi^k, \phi^k)_N^k$$

$$+ \sum_{j=2}^{N-2} w_j [\frac{\partial^2 \psi^k}{\partial n^2}(0, \xi_j) \frac{\partial \phi^k}{\partial n}(0, \xi_j) - \frac{\partial^3 \psi^k}{\partial n^3}(0, \xi_j) \phi^k(0, \xi_j)] ,$$

for $k = 1, 2$.

Theorem 6.2 *If there is an element $\Psi_N^b \in V_N$ satisfying (80) and (81) then the variational problem: Find $\Psi_N \in V_N$ such that $(\Psi_N - \Psi_N^b) \in V_{N,0}$ and*

$$a_N(\Psi_N, \Phi_N) = (f^1, \phi^1)_N^1 + (f^2, \phi^2)_N^2 , \tag{83}$$

for all $\Phi = (\phi^1, \phi^2) \in V_{N,0}$, is equivalent to the collocation problem: Find $\Psi_N \in V_N$ such that

$$\nabla^4 \psi_N^k(x, y) = f^k(x, y) , \qquad (x, y) \in R_k , \quad k = 1, 2 , \tag{84}$$

$$\psi_N^k(x, y) = g_1(x, y) , \qquad (x, y) \in S_k \cup T_k , \quad k = 1, 2 , \tag{85}$$

$$\frac{\partial \psi_N^k}{\partial n}(x, y) = g_2(x, y) , \qquad (x, y) \in S_k \cup T_k , \quad k = 1, 2 , \tag{86}$$

$$\frac{\partial^2 \psi_N^k}{\partial n \partial t}(x, y) = \frac{\partial g_2}{\partial t}(x, y) , \qquad (x, y) \in T_k , \quad k = 1, 2 , \tag{87}$$

$$\frac{\partial^3}{\partial x^3}(\psi_N^2 - \psi_N^1)(0, \xi_j) = -w_{N-1}^1 [(\nabla^4 \psi_N^1 - f)(0, \xi_j)]$$

$$-w_1^2 [(\nabla^4 \psi_N^2 - f)(0, \xi_j)]$$

$$-\overline{w}_{N-1}^1 [\frac{\partial}{\partial x}(\nabla^4 \psi_N^1 - f)(0, \xi_j)]$$

$$-\overline{w}_1^2 [\frac{\partial}{\partial x}(\nabla^4 \psi_N^2 - f)(0, \xi_j)] , \quad 2 \leq j \leq N-2 , \tag{88}$$

$$\frac{\partial^2}{\partial x^2}(\psi_N^2 - \psi_N^1)(0, \xi_j) = w_{N-1}^1 [(\nabla^4 \psi_N^1 - f)(0, \xi_j)]$$

$$+ w_1^2 [(\nabla^4 \psi_N^2 - f)(0, \xi_j)] , \quad 2 \leq j \leq N-2 , \tag{89}$$

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where

$$R_k = \{(\xi_i^k, \xi_j) : 2 \le i, j \le N - 2\}, k = 1, 2,$$
$$S_1 = \{(\xi_i^1, \pm 1), (-1, \xi_i) : 2 \le i \le N - 2\},$$
$$S_2 = \{(\xi_i^2, \pm 1), (1, \xi_i) : 2 \le i \le N - 2\},$$
$$T_1 = \{(-1, \pm 1), (0, \pm 1)\},$$
$$T_2 = \{(0, \pm 1), (1, \pm 1)\},$$

and

$$\xi_i^1 = (\xi_i - 1)/2, \quad \xi_i^2 = (1 + \xi_i)/2, \quad 2 \le i \le N - 2.$$

Proof. We can show $(\Psi_N - \Psi_N^b) \in V_{N,0}$ as in the proof of Theorem 5.1. If we now choose as our test functions the following:

$$
\begin{cases}
\phi^1(x,y) = h_k(2x + 1)h_l(y), \quad \phi^2(x,y) = 0, & 2 \le k, l \le N - 2, \\
\phi^1(x,y) = 0, \quad \phi^2(x,y) = h_k(2x - 1)h_l(y), & 2 \le k, l \le N - 2, \\
\phi^1(x,y) = h_{N-1}(2x + 1)h_l(y), \quad \phi^2(x,y) = h_1(2x - 1)h_l(y), & 2 \le l \le N - 2, \\
\phi^1(x,y) = \overline{h}_{N-1}(2x + 1)h_l(y), \quad \phi^2(x,y) = \overline{h}_1(2x - 1)h_l(y), & 2 \le l \le N - 2,
\end{cases}
\tag{90}
$$

then we obtain immediately (84), (85), (88) and (89). Conversely, since these $2(N-3)(N-2)$ test functions constitute a basis for $V_{N,0}$, (84)-(89) implies (83).

7 Numerical Results

The quadrature rule (8) is used to compute approximations to the integrals

(a) $\int_{-1}^1 w_\lambda(x) e^x \cos(\pi x) dx,$

(b) $\int_{-1}^1 w_\lambda(x) x e^x dx,$

when $\lambda = 0$ and $\lambda = -1/2$. The errors in the quadrature rule are given in Tables 3 and 4 for integrals (a) and (b), respectively, for different values of N. The quadrature rule evaluates the integrals accurate to machine precision for a value of N as small as 17.

7.1 1-D Problems

Numerical solutions to the fourth-order model problem (1) are obtained when the exact solution is given by

(a) $u(x) = (1 - x^2)^2 \sin(\pi x),$

(b) $u(x) = 1 + \sin(2\pi x).$

In example (a) the boundary conditions are homogeneous whereas for (b) we have inhomogeneous boundary conditions. The differential equation is collocated at the generalized Legendre and Chebyshev nodes given by the zeros of $(1 - x^2)P''_{N-1}(x)$ and $(1 - x^2)T''_{N-1}(x)$,

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respectively. The error in the numerical solution is measured using weighted norms based on the corresponding generalized quadrature rule. The infinity norm is also given to show the maximum pointwise error at the collocation points. These are displayed in Tables 5 and 6 for examples (a) and (b), respectively, where we define

$$\| e \|_{2,w} = [\sum_{j=1}^{N-1} w_j e_j^2 + \overline{w}_1((e_1')^2 - (e_{N-1}')^2))]^{1/2},$$

$$\| e \|_\infty = \max_{1 \le j \le N-1} | e_j | ,$$

and

$$e_j = u(x_j) - u_N(x_j) , \quad j = 1, 2, ..., N-1 ,$$

where the points $x_j, 1 \le j \le N-1$, are the generalized nodes. The usual exponential convergence of spectral approximations to smooth solutions of differential equations is observed with accuracy to machine precision being obtained when $N = 24$.

Next we apply these techniques in the case of domain decomposition. For simplicity we consider a partition of the interval $[-1,1]$ into the two subintervals $[-1,0]$ and $[0,1]$ with common point $x = 0$. We solve again the model problems (a) and (b) using the collocation scheme (59)-(61). The corresponding error norms are shown in Tables 7 and 8, respectively. The mono-domain and two-domain spectral approximations converge exponentially as expected. The two-domain approximation converges slower than the mono-domain approximation for the same total number of collocation points since for the problems considered here there is no particular advantage to be gained in using the former since the solutions are smooth and the problem is one-dimensional. Patera (1984) observes similar behaviour for spectral element approximations to second-order problems. The power and usefulness of a multi-domain approach for pseudospectral methods will be demonstrated for problems defined in nonrectangular geometries in 2-D.

7.2 2-D Problems

Numerical solutions to the biharmonic equation are obtained using the pseudospectral method when the exact solution is given by

(a) $\psi(x,y) = (1 - x^2)^2(1 - y^2)^2 sin(\pi y)$,

(b) $\psi(x,y) = (1 - x^2)^2(1 - y^2)^2 sin(\pi x) sin(\pi y)$,

(c) $\psi(x,y) = sin(2\pi x) sin(2\pi y)$.

In examples (a) and (b) the boundary conditions are homogeneous whereas for (c) the Neumann boundary condition is inhomogeneous. The mixed second order derivative ψ_{xy} is zero at the four corners of Ω for these three model problems. The biharmonic equation is collocated at the Cartesian product of generalized Legendre nodes. The weighted and infinity norms of the errors are shown in Table 9 for problems (b) and (c). We see that a numerical solution correct to machine accuracy is obtained on a grid as coarse as 21×21.

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In the case of domain decomposition we consider a partition of Ω into two rectangular subdomains $[-1, 0] \times [-1, 1]$ and $[0, 1] \times [-1, 1]$ with common interface $x = 0$. Here we solve the model problem (a) using the collocation scheme (84)-(89). Both the mono-domain and two-domain pseudospectral approximations converge exponentially as expected. Again the two-domain approximation converges slower than the mono-domain approximation for the same total number of collocation points. This phenomenon was observed for 1-D problems too. In Figure 1 we give the contours of the approximation to the solution of problem (a) obtained using domain decomposition with $N = 20$. This figure is included to show the smoothness of the contours across the interface $x = 0$.

7.3   2-D Problems in Nonrectangular Domains

We extend the ideas developed in this paper to the solution of the Stokes problem for the flow through an L-shaped channel. The flow geometry in this example is nonrectangular for which a standard single domain pseudospectral approximation is not applicable. The ability of pseudospectral methods to solve problems in this kind of geometry justifies the development of the theory of the multi-domain formulation considered earlier. The flow domain is divided into three rectangular subdomains as shown in Fig. 2. The stream function within each subdomain is approximated by a pseudospectral representation which interpolates values of the stream function at interior collocation points and values of the stream function and its normal derivative on the boundaries and subdomain interfaces. These representations are C^1 continuous across the subdomain interfaces. The unknowns in the pseudospectral approximations are determined from the collocation scheme derived from the discrete variational formulation. This scheme results in C^3 continuous approximations asymptotically.

If approximations of degree N are used in each direction in each subdomain then the collocation equations yield a system of $(3N - 5)(N - 3)$ equations for the $(3N - 5)(N - 3)$ unknowns. A total of $2(N - 3)$ of these unknowns represent the values of the normal derivatives of ψ at the interior nodes along the interfaces between subdomains Ω_1 and Ω_2 and between subdomains Ω_2 and Ω_3. The remaining unknown values are the nodal values of ψ at the interior and interface points of subregions Ω_1, Ω_2 and Ω_3. The collocation equations give rise to a linear algebraic system $Au = b$. The vector u contains the nodal values of ψ and also the normal derivative of ψ at the interface nodes. The block tridiagonal structure of the matrix A for the L-shaped domain is shown in Fig. 3. This system is solved using a Crout factorization subroutine from the NAG Library (1988). A more efficient direct solution technique which takes account of the inherent matrix structure is the almost block diagonal solver of Brankin and Gladwell (1990) which has been used in spectral calculations by Karageorghis and Phillips (1990,1991). However, this subroutine has not yet been incorporated into the present algorithm.

The entry and exit lengths, a and b respectively, are chosen to be long enough to obtain fully developed flow. In Figs. 4 and 5 we show the contours of the stream function for $N = 14$, $b = 7$, $c = 1$ with $a = -3$ and $a = -5$, respectively. A small weak vortex is observed in the salient corner. Fully developed flow is reached within a channel width of the reentrant corner.

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8 Conclusions

Pseudospectral approximations to the solution of fourth-order elliptic partial differential equations are constructed using a collocation procedure based on the nodes of generalized Gaussian quadrature rules. Analytic expressions for the weights appearing in these quadrature rules are derived and their forms for the generalized Legendre and Chebyshev rules are given. The equivalence between a discrete variational form of the differential problem with suitably defined inner products and a collocation scheme is demonstrated when the collocation points are chosen to be the zeros of certain ultraspherical polynomials. The natural choice of collocation points for fourth-order problems differs from the choice for second-order problems, viz. the Gauss-Lobatto points. The usual convergence properties of spectral approximations are observed.

A domain decomposition procedure based on the generalized Gauss-Legendre nodes is considered. Pseudospectral approximations which are automatically C^1- continuous at the subinterval interfaces are used to represent the solution. An examination of the corresponding discrete variational problem results in an equivalent collocation method. The resulting approximation is shown to be C^3- continuous at the interfaces asymptotically, i.e. as the order of the approximations is increased in each subinterval. The scheme is analyzed and an error estimate is derived for the domain decomposed problem.

For fourth-order problems in two dimensions we propose using a tensor product of the one-dimensional basis functions to represent the solution. The equivalence between the collocation method defined by collocating the differential equation on a grid formed by the tensor product of the one-dimensional collocation points and a discrete variational formulation of the problem is described as well as the corresponding domain decomposition problem. It is intended to apply this collocation method to the solution of the Navier-Stokes equations in rectangularly decomposable domains using a stream function formulation even though a simple variational principle does not exist for these equations.

An application of this methodology to a biharmonic problem in a nonrectangular geometry is described. A single domain approach is not feasible for this class of problems unless one first transformed the original irregular domain to a simpler rectangular one. However, this would be cumbersome if it could be done at all since a transformation would need to be found for each new geometry.

TABLE 1

Extreme eigenvalues of A

N	λ_{min}	λ_{max}	λ_{max}/N^8
8	0.3128+2	0.4768+4	0.2842-3
12	0.3128+2	0.1091+6	0.2537-3
16	0.3128+2	0.1111+7	0.2587-3
20	0.3128+2	0.6788+7	0.2652-3
24	0.3128+2	0.2979+8	0.2706-3
28	0.3128+2	0.1039+9	0.2750-3
32	0.3128+2	0.3065+9	0.2788-3

TABLE 2

Extreme eigenvalues of $H^{-1}A$

N	λ_{min}	λ_{max}
8	1.000	3.312
12	1.000	4.180
16	1.000.	4.635
20	1.000	4.915
24	1.000	5.104
28	1.000	5.241
32	1.000	5.344

TABLE 3

Quadrature error in the approximation of $\int_{-1}^{1} w(x)e^x \cos(\pi x)dx$ for different weight functions

N	$w(x) = 1$	$w(x) = (1 - x^2)^{-1/2}$
5	0.497 -2	0.689 -2
9	0.767 -10	0.122 -8
17	0.300 -15	0.710 -14

TABLE 4

Quadrature error in the approximation of $\int_{-1}^{1} w(x)xe^x dx$ for different weight functions

N	$w(x) = 1$	$w(x) = (1 - x^2)^{-1/2}$
5	0.579 -5	0.843 -5
9	0.800 -14	0.640 -14
17	0.300 -15	0.360 -14

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TABLE 5

Errors in the numerical solution of the model problem (1)
with exact solution given by $u(x) = (1 - x^2)^2 sin(\pi x)$

N	$\lambda = 0$ $\| e \|_{2,w}$	$\lambda = 0$ $\| e \|_\infty$	$\lambda = -1/2$ $\| e \|_{2,w}$	$\lambda = -1/2$ $\| e \|_\infty$
8	1.387-2	1.515-2	3.041-2	3.200-2
12	2.941-6	2.954-6	2.057-5	2.783-5
16	3.077-10	3.041-10	6.355-9	7.289-9
24	1.177-13	1.329-13	1.804-13	2.018-13

TABLE 6

Errors in the numerical solution of the fourth-order problem
with $u(\pm 1) = 1$, $du/dx(\pm 1) = 2\pi$
and exact solution $u(x) = 1 + sin(2\pi x)$

N	$\lambda = 0$ $\| e \|_{2,w}$	$\lambda = 0$ $\| e \|_\infty$	$\lambda = -1/2$ $\| e \|_{2,w}$	$\lambda = -1/2$ $\| e \|_\infty$
8	0.236	0.266	0.450	0.494
12	1.883-4	1.845-4	5.926-4	7.976-4
16	1.181-7	1.150-7	8.848-7	1.052-6
24	5.795-13	6.701-7	4.534-13	5.190-13

TABLE 7

Errors in the numerical solution of the model problem (1) with exact solution given by
$u(x) = (1 - x^2)^2 sin(\pi x)$ using domain decomposition

N	$\lambda = 0$ $\| e \|_{2,w}$	$\lambda = 0$ $\| e \|_\infty$	$\lambda = -1/2$ $\| e \|_{2,w}$	$\lambda = -1/2$ $\| e \|_\infty$
6	2.633-2	2.024-2	3.294-2	2.747-2
8	1.490-3	1.096-3	1.890-3	1.377-3
12	3.494-7	2.642-7	4.075-7	3.073-7
16	1.293-11	1.243-11	1.254-11	9.523-11

TABLE 8

Errors in the numerical solution of the fourth-order problem
with $u(\pm 1) = 1$, $du/dx(\pm 1) = 2\pi$ and exact solution $u(x) = 1 + sin(2\pi x)$ using domain decomposition

N	$\lambda = 0$ $\| e \|_{2,w}$	$\lambda = 0$ $\| e \|_{\infty}$	$\lambda = -1/2$ $\| e \|_{2,w}$	$\lambda = -1/2$ $\| e \|_{\infty}$
6	0.326	0.230	0.476	0.331
8	2.794-2	2.080-2	3.580-2	2.632-2
12	3.221-5	2.430-5	3.757-5	2.839-5
16	7.491-9	5.677-9	7.961-9	6.058-9

TABLE 9

Errors in the numerical solution of the biharmonic problem (63)-(65)
with exact solutions given by (b) and (c)

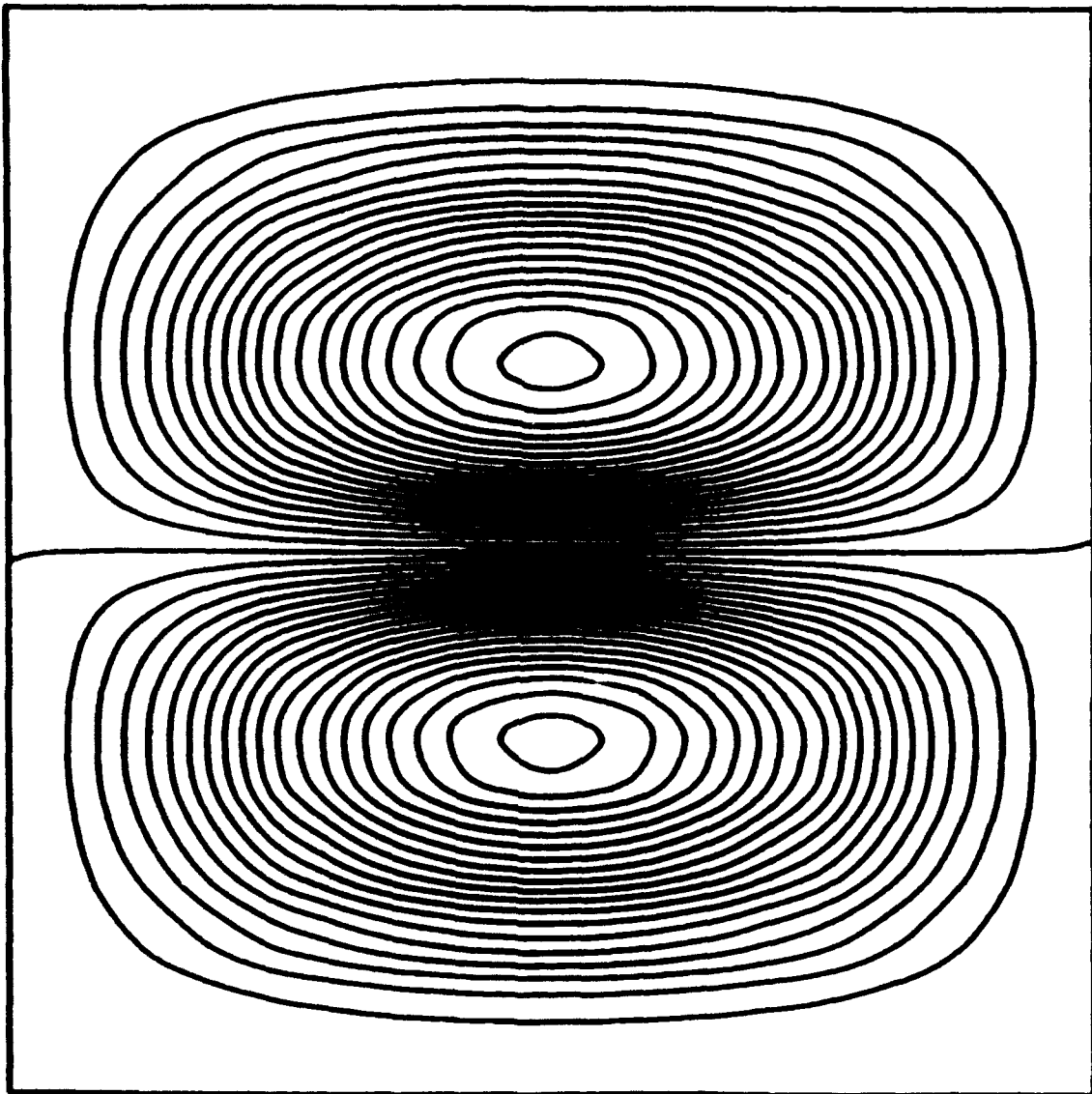| N	Problem (b) $\| e \|_{2,w}$	Problem (b) $\| e \|_{\infty}$	Problem (c) $\| e \|_{2,w}$	Problem (c) $\| e \|_{\infty}$
6	0.347	0.512	1.467	2.483
8	5.145-3	3.779-2	5.831-2	8.238-2
12	4.667-6	9.413-5	1.879-4	4.925-4
16	5.631-10	7.836-9	3.856-8	7.437-8
20	2.935-13	4.291-12	9.648-11	1.502-10

Figure 1. Contour plots of $\psi(x, y)$ for problem (a) when $N = 20$, using domain decomposition and the generalized Legendre pseudospectral method.
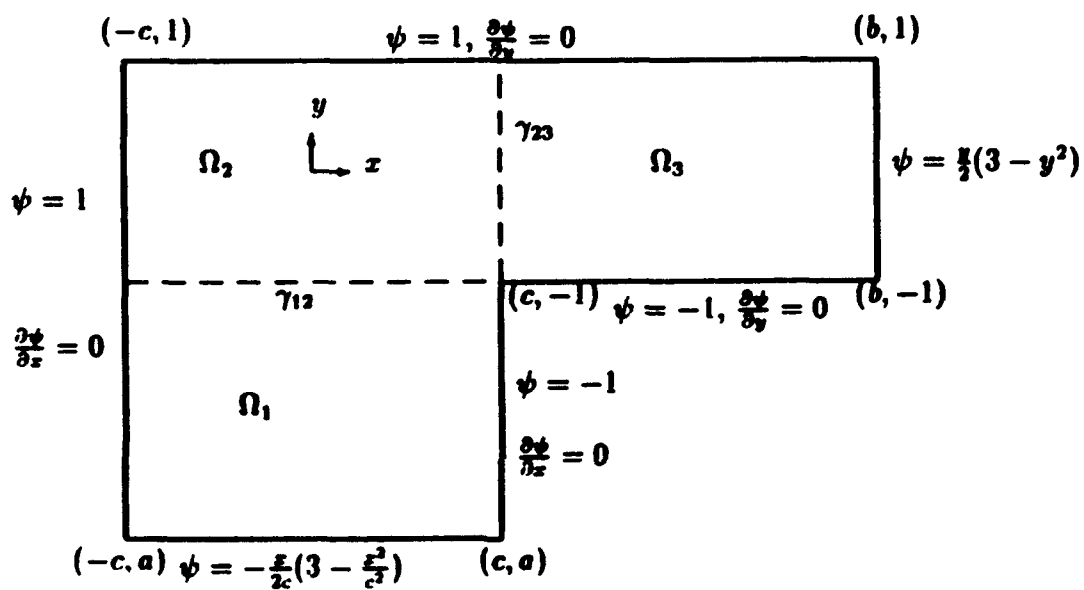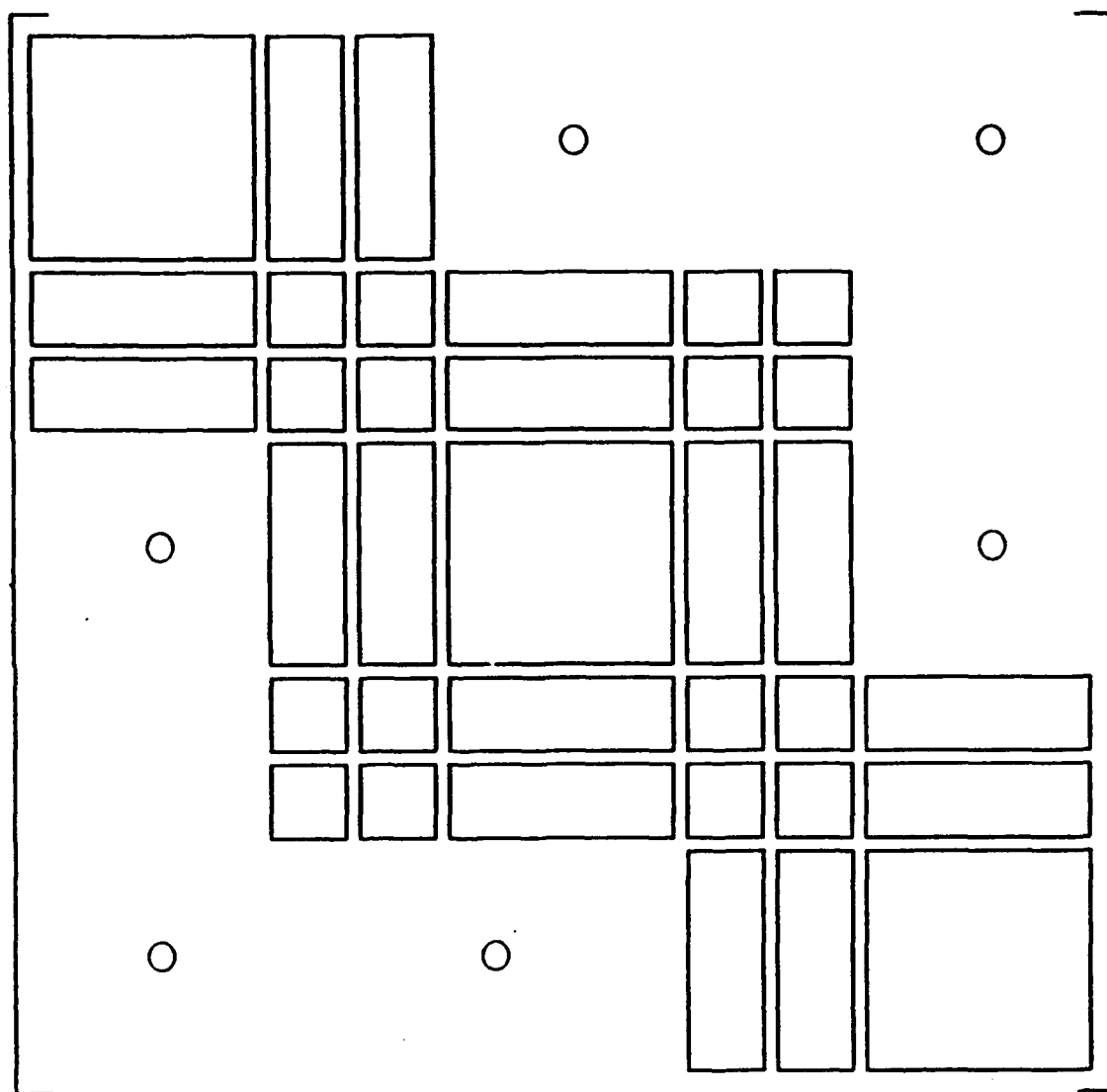
Figure 2. The L-shaped domain and boundary conditions.

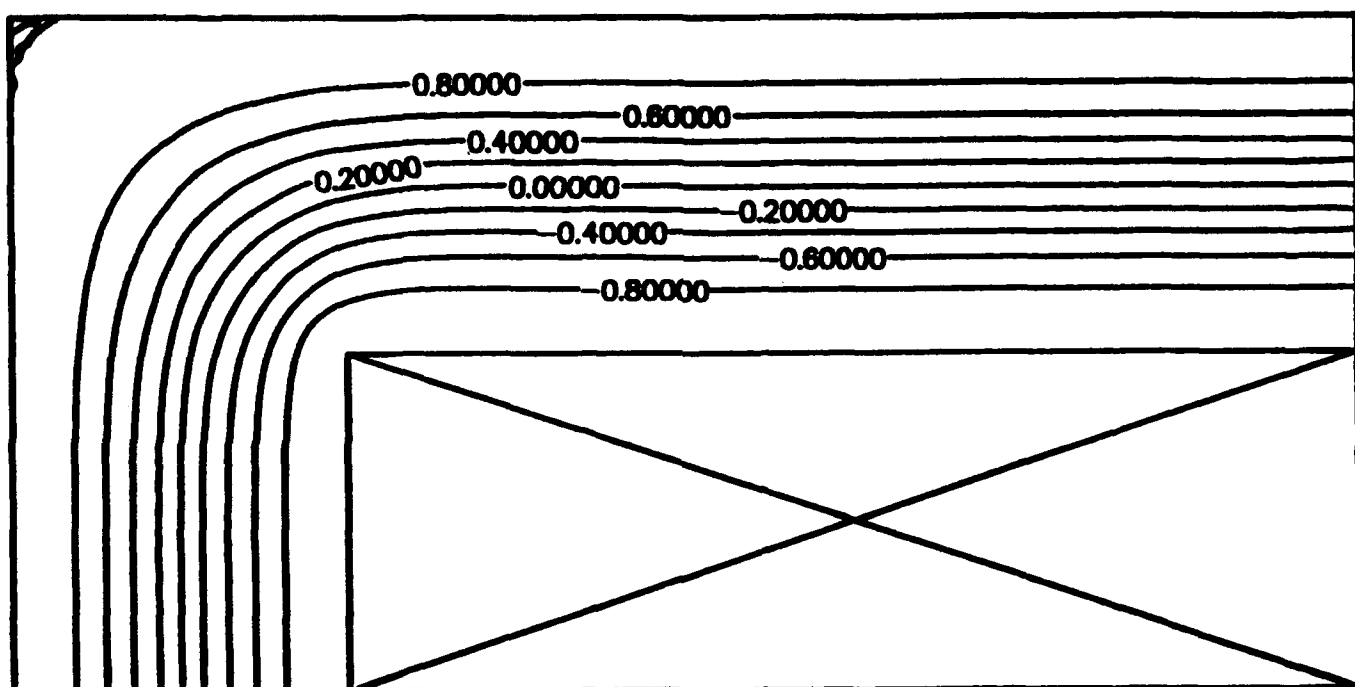Figure 3. Structure of the matrix A for the domain decomposition problem in 2-D.

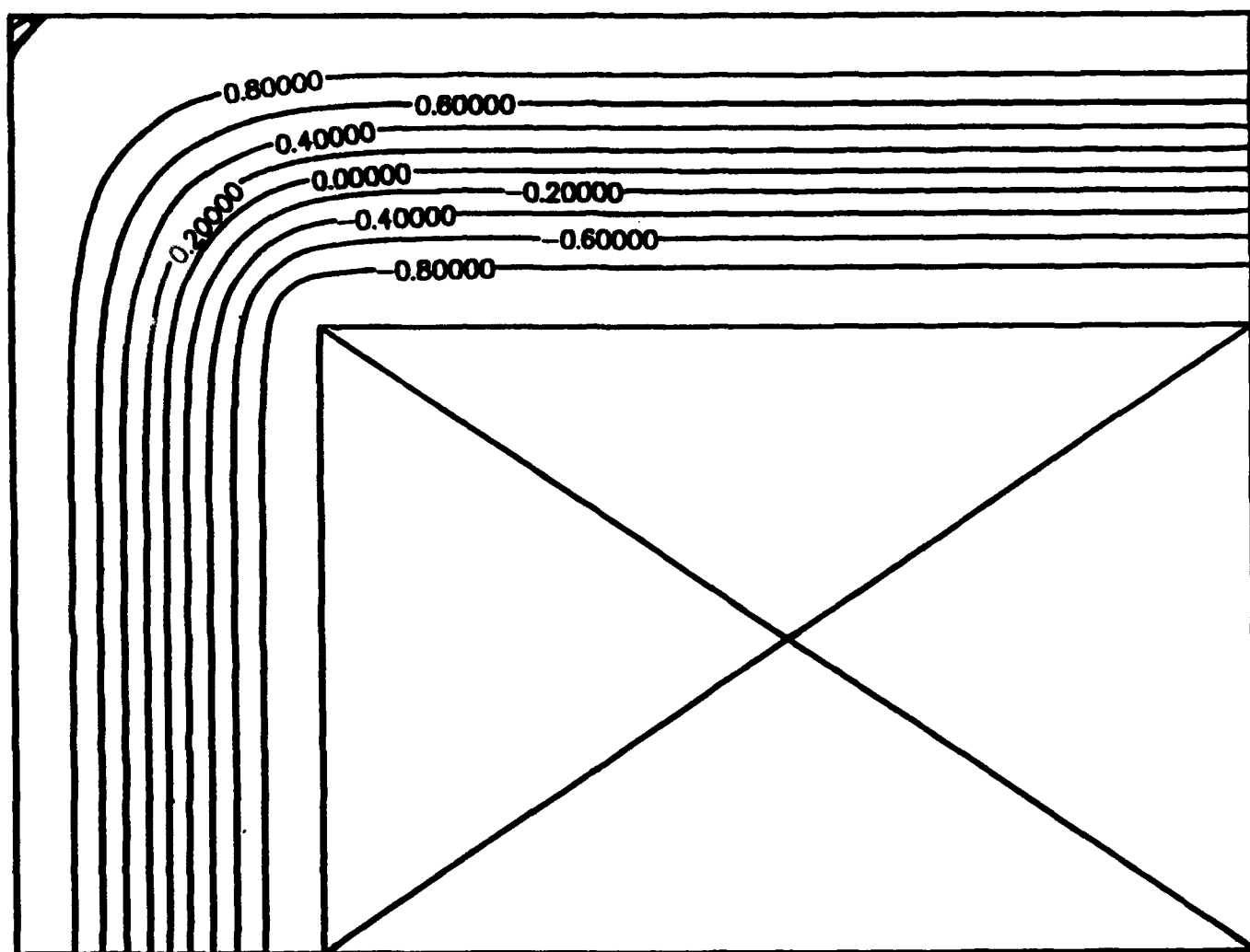Figure 4. Contours of $\psi(x,y)$ for $a = -3$, $b = 7$, $c = 1$ and $N = 16$.

Figure 5. Contours of $\psi(x, y)$ for $a = -5$, $b = 7$, $c = 1$

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A Properties of Ultraspherical Polynomials

The ultraspherical or Gegenbauer Polynomials are the solutions of the differential equation

$$[w_{\lambda+1}(x)G_n^{(\lambda)\prime}(x)]' + n(n + 2\lambda + 1)w_\lambda(x)G_n^{(\lambda)}(x) = 0, \tag{1}$$

that are bounded at $x = \pm 1$, where

$$w_\lambda(x) = (1 - x^2)^\lambda, \quad \lambda > -1. \tag{2}$$

They are orthogonal with respect to $w_\lambda(x)$ over the interval [-1,1] :

$$\int_{-1}^1 w_\lambda(x)G_m^{(\lambda)}(x)G_n^{(\lambda)}(x)dx = \gamma_m \delta_{m,n} , \tag{3}$$

where

$$\gamma_m = \frac{2^{2\lambda+1}[\Gamma(m + \lambda + 1)]^2}{(2m + 2\lambda + 1)m!\Gamma(m + 2\lambda + 1)} , \tag{4}$$

and Γ is the gamma function. At $x = \pm 1$, $G_n^{(\lambda)}(x)$ satisfies the condition

$$G_n^{(\lambda)}(\pm 1) = (\pm 1)^n \frac{\Gamma(n + \lambda + 1)}{n!\Gamma(\lambda + 1)}. \tag{5}$$

The ultraspherical polynomials may be generated using the recurrence relation

$$
\begin{aligned}
(n &+ 1)(n + 2\lambda + 1)G_{n+1}^{(\lambda)} \\
&= (2n + 2\lambda + 1)(n + \lambda + 1)xG_n^{(\lambda)} - (n + \lambda)(n + \lambda + 1)G_{n-1}^{(\lambda)}, \\
G_0^{(\lambda)}&(x) = 1 \quad , \quad G_1^{(\lambda)}(x) = (\lambda + 1)x.
\end{aligned}
\tag{6}
$$

The leading coefficient, A_n, of $G_n^{(\lambda)}(x)$ is given by

$$A_n = \frac{1}{2^n}\frac{\Gamma(2n + 2\lambda + 1)}{n!\Gamma(n + 2\lambda + 1)}. \tag{7}$$

We have the following integrals involving ultraspherical polynomials (Erdelyi (1954), p.284) :

$$\int_{-1}^1 (1 - x)^\lambda(1 + x)^\sigma G_n^{(\lambda)}(x)dx = \frac{2^{\lambda+\sigma+1}\Gamma(\sigma + 1)\Gamma(\lambda + n + 1)\Gamma(\sigma - \lambda + 1)}{n!\Gamma(\sigma - \lambda - n + 1)\Gamma(\lambda + \sigma + n + 2)}, \tag{8}$$

$$\int_{-1}^1 (1 - x)^\sigma(1 + x)^\lambda G_n^{(\lambda)}(x)dx = \frac{2^{\lambda+\sigma+1}\Gamma(\sigma + 1)\Gamma(\lambda + n + 1)\Gamma(\lambda - \sigma + n)}{n!\Gamma(\lambda - \sigma)\Gamma(\lambda + \sigma + n + 2)}, \tag{9}$$

where $\lambda, \sigma > -1$.

The ultraspherical polynomials satisfy the recursion relation

$$(1 - x^2)G_n^{(\lambda)\prime}(x) = -nxG_n^{(\lambda)}(x) + (n + \lambda)G_{n-1}^{(\lambda)}(x). \tag{10}$$

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13. ABSTRACT *(Maximum 200 words)*
Collocation schemes are presented for solving linear fourth order differential equations in one and two dimensions. The variational formulation of the model fourth order problem is discretized by approximating the integrals by a Gaussian quadrature rule generalized to include the values of the derivative of the integrand at the boundary points. Collocation schemes are derived which are equivalent to this discrete variational problem. An efficient preconditioner based on a low-order finite difference approximation to the same differential operator is presented. The corresponding multi-domain problem is also considered and interface conditions are derived. Pseudospectral approximations which are C^1 continuous at the interfaces are used in each subdomain to approximate the solution. The approximations are also shown to be C^3 continuous at the interfaces asymptotically. A complete analysis of the collocation scheme for the multi-domain problem is provided. The extension of the method to the biharmonic equation in two dimensions is discussed and results are presented for a problem defined in a non-rectangular domain.

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